Fixed Points In Mathematics

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Abstract

Fixed points appear in various branches of Mathematics. In this research paper we have mentioned some important branches of Mathematics in which fixed points play a very important role. Precisely we have considered number theory, numerical analysis, complex analysis, linear algebra and transformational geometry and give some important mappings and also compute their fixed points.

Keywords: Fixed point, Fermat's theorem, Möbius transformation, exponential mapping, PageRank

2000 Mathematics Subject Classification:47H10

1 Introduction

Fixed points appear in Mathematics and play a crucial role to establish many useful results. For example, the celebrated Fermat's theorem in number theory can be easily proved by simply computing the number of fixed points of a mapping. The internet search engines like Google use fixed point results to decide the priority of web pages to be displayed at the top. Fixed points and the results on fixed points have many applications in engineering, science, social sciences like economics, computer science etc. This research paper is a survey of the fields where the fixed points are useful.

Definition 1.1. *[8] Let X be a set and $T : X \to X$ be a map. A point $x \in X$ is called a fixed point of T if $T(x) = x$.*

2 Fixed Points in Number Theory

It is well known that study of prime numbers in number theory has taken giant leap after the invention of celebrated theorem due to Fermat. It is well known as Fermat's little theorem and stated as follows:

Theorem 2.1. *If q is a prime then $k^q \equiv k (mod\ q)$ for all positive integers k.*

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Fermat;s little theorem and it's corollaries are applicable to prove the results about periodic points in dynamical systems [[1].[3]]. Various proofs of the theorem are available in literature. For instance see [[2],[4],[5]]. We observe below that Fermat's theorem is a simple consequence of counting the fixed points of a function defined below.

Definition 2.2. *[6] The function* $T_k : [0,1] \to [0,1]$ *is defined as follows:*

$$T_k x = \begin{cases} kx & \text{if, } 0 \le x \le \frac{1}{k} \\ kx - i & \text{if, } \frac{i}{k} < x \le \frac{i+1}{k}, \end{cases}$$

where $1 \le i \le k-1$ *and* $k \ge 2$ *.*

Example 2.3. *If* $k = 4$ *, the function in the definition 2.2 becomes*

$$Tx = \begin{cases} 4x & \text{if, } 0 \le x \le \frac{1}{4} \\ 4x - 1 & \text{if, } \frac{1}{4} < x \le \frac{1}{2}, \\ 4x - 2 & \text{if, } \frac{1}{2} < x \le \frac{3}{4}, \\ 4x - 3 & \text{if, } \frac{3}{4} < x \le 1. \end{cases}$$

The graph of this function is given below.

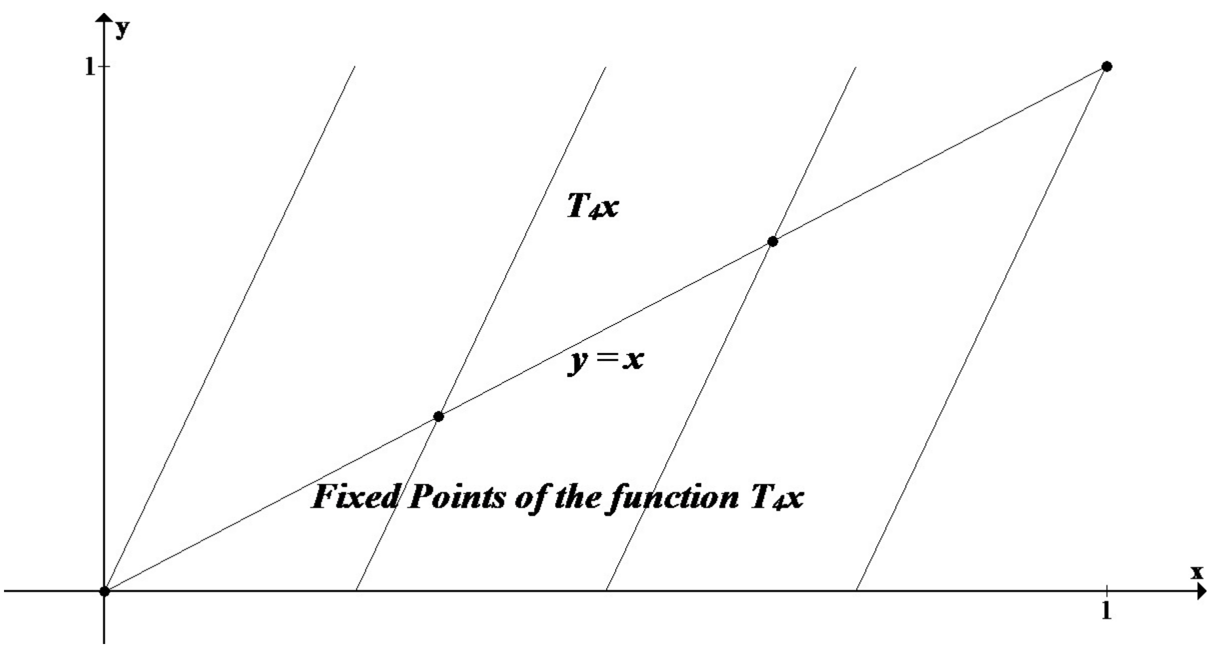

Figure 1: The line $y = x$ intersects the graph of $T_4(x)$ four times.

The function $T_4(x)$ has four fixed points. In general we have the following lemma.

Lemma 2.4. *[6] The function* $T_k(x)$ *has* k *number of fixed points, the function* $T_k^2(x) = (T_k \circ T_k)(x)$ *has* k^2 *number of fixed points and in general the function* $T_k^n(x) = (T_k \circ T_k \circ \cdots \circ T_k(n\ times))(x)$ *has* k^n *fixed points.*

Definition 2.5. *[6] The fixed points of* $T_k(x)$ *of period* n *is a fixed point* $x \in [0,1]$ *for which* $T_k^n(x) = x$ *. Thus the fixed point of* $T_k(x)$ *of period* n *is a fixed point of* $T_k^n(x)$ *. Then* n *is called as a period of* x *.*

Definition 2.6. *[6] A minimal period of a fixed point* $x \in [0,1]$ *is the value of* n *such that* $T_k^l x \neq x$ *for all* $0 < l < n$ *and* $T_k^n x = x$ *.*

We denote by $N_n(T_k)$ the number of fixed points of minimal period n of the function T_k .

Definition 2.7. *[6] For each point* $x \in [0,1]$ *, we define orbit of* x *to be the set* $\{x, T_k x, T_k^2 x, \cdots\}$ *.*

If $x \in [0,1]$ has a period n , then the orbit of x contains at most n distinct points. Such an orbit is called n -cycle.

Definition 2.8. *[6] If* T *has minimal period* n *, then the orbit of* x *contains precisely* n *distinct elements:* $x, T_k x, T_k^2 x, \cdots, T_k^{n-1} x$ *. Such orbits are called minimal* n *-cycles.*

Lemma 2.9. *[6] The following hold:*
1) If $x_0 \in [0,1]$ *is a fixed point of period* n *that has minimal period* m *, then* $m \mid n$ *. That is* n *is divisible by* m *(or* m *divides* n *).*
2) Minimal m *-cycles are either mutually identical or disjoint. That is any two minimal* m *-cycles are either have all elements common or no element common.*
3) For all $n \geq 1, n \mid N_n$ *, provided* N_n *is finite. That is* N_n *is divisible by* n *.*

The following lemma is useful in the next theorem.

Lemma 2.10. *[6] The following hold:*
1) The function $T_k x$ *defined in the definition 2.2 has* k^n *fixed points of period* n *.*
2) For all integers $k > 1$ *and all integers* $n \geq 1$ *,* $k^n = \sum_{m|n} N_m(T_k)$ *.*

Now we give a simple proof of Fermat's little theorem by using fixed point technique in the following theorem.

Theorem 2.11. *For all integers* $k \geq 2$ *and all primes* q *,* $k^q \equiv k(mod\ q)$ *.*

Proof:[6] We know that if n is a prime integer then the only divisors of n are 1 and n itself. By the above lemma 2.10

$$k^q = \sum_{m|q} N_m(T_k) = N_1 + N_q.$$

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Now

$$\begin{aligned} N_1 &= Number\ of\ fixed\ points\ of\ T_k\ of\ period\ 1 \\ &= Number\ of\ fixed\ points\ of\ T_k^1 \\ &= T_k \\ &= k. \end{aligned}$$

Hence, $k^q = k + N_q$. That is $k^q - k = N_q$. But N_q is divisible by q by lemma 2.9. Thus $k^q \equiv k(mod\ q)$.

Thus Fermat's theorem follows from the consequence of counting the fixed points of $T_k^q(x) = (T_k \circ T_k \cdots \circ T_k)(q - times)(x)$

3 Fixed Points in Numerical Analysis

In science and engineering studies, it is frequently required to find the roots of the equation of the form $f(x) = 0$. Hence solving the equation $f(x) = 0$ for x is one of the central problems in numerical analysis. A root of the equation $f(x) = 0$ is the value r of x such that $f(r) = 0$. The following theorem explains the use of fixed points in solving this problem.

Theorem 3.1. *r is a root of the function $f(x)$ if and only if r is a fixed point of the function $T(x) = x - f(x)$.*

Proof: Suppose r is root of the function $f(x)$. Then we get $f(r) = 0$. Therefore $T(r) = r - f(r) = r - 0 = r$. Thus r is fixed point of T. Conversely, suppose r is a fixed point of T. Thus $T(r) = r$. That is $r - f(r) = r$. Thus $f(r) = 0$. So r is a root of the function $f(x)$. Hence the theorem is proved.

Thus in order to find a root of the equation $f(x) = 0$, one can find the fixed points of the function $T(x) = x - f(x)$.

Example 3.2. *To find the root of the function $f(x) = x^3 - 6x^2 + 11x - 6$, we consider the function $T(x) = x - f(x) = -x^3 + 6x^2 - 10x + 6$. We observe that $x = 1, 2, 3$ are the three fixed point of the function $T(x)$.*
$T(1) = -1^3 + 6(1^2) - 10(1) + 6 = -1 + 6 - 10 + 6 = 1$
$T(2) = -2^3 + 6(2^2) - 10(2) + 6 = -8 + 24 - 20 + 6 = 2$
$T(3) = -3^3 + 6(3^2) - 10(3) + 6 = -27 + 54 - 30 + 6 = 3$
Hence $x = 1, 2, 3$ are the roots of the function $f(x) = x^3 - 6x^2 + 11x - 6$.

To find the fixed points of the function $Tx = x - f(x)$, the iteration method may be used. This method is described as follows:

Step 1. Choose a point x_0 that is approximately near to the fixed point of Tx.
Step 2. Compute $x_1 = Tx_0, x_2 = Tx_1, \cdots, x_n = Tx_{n-1}, \cdots$.
Step 3. If the sequence $\{x_n\}_{n=0}^{\infty}$ converges then it has the limit as a fixed point of Tx.

4 Fixed Points in Complex Analysis

In this section we consider two important types of functions:
1] Möbius transformations,
2] Exponential mappings.
We study these functions and mention important results about their fixed points.

Definition 4.1. *[7] A transformation $T : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ defined by*

$$T(z) = \frac{a_1 z + a_2}{a_3 z + a_4}$$

where $a_1, a_2, a_3, a_4 \in \mathbb{R}, a_1 a_4 - a_2 a_3 > 0$ is called a Möbius transformation.

The following theorem describes the nature of fixed points of a Möbius transformation.

Theorem 4.2. *[7] A non-identity Möbius transformation has at most two fixed points.*

Proof: Case 1. If $a_1 = a_4 = 1, a_2 = a_3 = 0$, then the Möbius transformation becomes $T(z) = z$. This is an identity transformation. In this case all the points in $\mathbb{C} \cup \{\infty\}$ are fixed points of $T(z)$.
Case 2. Now suppose that the Möbius transformation is not an identity transformation. Suppose $z \neq 0$. We may write the Möbius transformation as:

$$T(z) = \frac{a_1 + \frac{a_2}{z}}{a_3 + \frac{a_4}{z}}.$$

Taking limit as $z \to \infty$ or $\frac{1}{z} \to 0$, gives

$$\lim_{z \to \infty} T(z) = \frac{a_1}{a_3}.$$

If $a_3 = 0$ then we get $\lim\limits_{z \to \infty} T(z) = \infty$. Thus $T(\infty) = \infty$. Therefore if $a_3 = 0$, then ∞ is a fixed point of the Möbius transformation.
Conversely, suppose that ∞ is a fixed point of a Möbius transformation. Thus

$$T(\infty) = \lim_{z \to \infty} \frac{a_1 + \frac{a_2}{z}}{a_3 + \frac{a_4}{z}}.$$

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Therefore $\frac{a_1}{a_3} = \infty$. Thus $a_3 = 0$. Hence a Möbius transformation has fixed point ∞ if and only if $a_3 = 0$.
Case 3. Suppose ∞ is a fixed point of a Möbius transformation. Then $a_3 = 0$. Then the Möbius transformation becomes

$$T(z) = \frac{a_1 z + a_2}{a_4} = \frac{a_1}{a_4} z + \frac{a_2}{a_4}.$$

Letting $z = \frac{a_2}{a_4 - a_1}$, we get

$$\begin{aligned} T\Big(\frac{a_2}{a_4 - a_1}\Big) &= \frac{a_1}{a_4}\frac{a_2}{a_4 - a_1} + \frac{a_2}{a_4} \\ &= \frac{a_2}{a_4}\Big\{\frac{a_1}{a_4 - a_1} + 1\Big\} \\ &= \frac{a_2}{a_4}\Big\{\frac{a_1 + a_4 - a_1}{a_4 - a_1}\Big\} \\ &= \frac{a_2}{a_4}\Big\{\frac{a_4}{a_4 - a_1}\Big\} \\ &= \frac{a_2}{a_4 - a_1} \end{aligned}$$

Thus if ∞ is a fixed point of a Möbius transformation, then $z = \frac{a_2}{a_4 - a_1}$ is also a fixed point of that Möbius transformation. Note that if $a_1 = a_4$ then that point is ∞ itself.
Case 4. Now suppose that ∞ is not a fixed point of a Möbius transformation $T(z)$. Then $a_3 \neq 0$. Thus $\frac{a_4}{a_3} \neq \infty$. Let z_0 be a fixed point of $T(z)$, that is $T(z_0) = z_0$. Now $z_0 \neq -\frac{a_4}{a_3}$, since if $z_0 = -\frac{a_4}{a_3}$, then

$$\begin{aligned} T(z_0) &= T\Big(-\frac{a_4}{a_3}\Big) \\ &= \frac{a_1\big(-\frac{a_4}{a_3}\big) + a_2}{a_3\big(-\frac{a_4}{a_3}\big) + a_4} \\ &= \infty \end{aligned}$$

Thus $z_0 = \infty$ and then $a_3 = 0$. This is a contradiction to $a_3 \neq 0$. Consider $a_3 z_0 + a_4$, which is not zero because $z_0 \neq -\frac{a_4}{a_3}$. Multiply by $a_3 z_0 + a_4$ to $T(z_0) = \frac{a_1 z_0 + a_2}{a_3 z_0 + a_4} = z_0$, to get

$$\begin{aligned} a_1 z_0 + a_2 &= z_0(a_3 z_0 + a_4) \\ \Rightarrow a_1 z_0 + a_2 &= a_3 z_0^2 + a_4 z_0 \\ \Rightarrow a_3 z_0^2 + (a_4 - a_1) z_0 - a_2 &= 0 \end{aligned}$$

This is a quadratic equation in z_0. This equation has either
1) two real solutions or

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2) one real solution or
3) two complex conjugate solutions.
These solutions are the fixed points of $T(z)$. Hence the proof is complete.
Consider now the fixed points of exponential function. It is the most commonly occurring function in engineering, biological sciences, physics, chemistry and astronomy. The derivative of an exponential function is proportional to its value. Therefore exponential function is important whenever the time rate of change of a quantity is proportional to its current value. Some important practical examples are rate of growth of certain bacteria, decay of radio active elements etc. Fixed points of a function is one of the prominent properties. So it is worth to consider fixed points of such an important exponential function. Following is the definition of an exponential function.

Definition 4.3. *[9] The complex exponential function $e_a : \mathbb{Z} \to \mathbb{Z}$ is defined as $e_a(z) = e^{az}$, where $e = 2.71828\cdots$ is an irrational number, and a is a real or complex constant.*

The constant a is normalized to 1 in many practical applications. The remaining part of this section is addressed to the existence and properties of fixed points of $e : \mathbb{Z} \to \mathbb{Z}$, defined by $e(z) = e^z$.

Definition 4.4. *[9] Fixed point of the exponential function $e(z) = e^z$ is a point $z \in \mathbb{Z}$ such that $e(z) = z$, that is $e^z = z$.*

Consider the following function also.

Definition 4.5. *[9] Define the function $e^- : \mathbb{Z} \to \mathbb{Z}$ by $e^-(z) = -e^z$*

In the context of the notation $e^-(z) = -e^z$ the function $e(z) = e^z$ may also be written as $e^+(z) = +e^z$.

Definition 4.6. *[9] Fixed point of the exponential function $e^-(z) = -e^z$ is a point $z \in \mathbb{C}$ such that $e^-(z) = z$ that is $-e^z = z$.*

Let z̄ be denote the complex conjugate of z. That is if z=x+iy, x,y $\in \mathbb{R}$ and i= $\sqrt{-1}$, then z̄=x-iy. Now

$$\begin{aligned} e(\overline{z}) &= e^{\overline{z}} \\ &= e^{x-iy} \\ &= e^x e^{-iy} \\ &= e^x(cosy - isiny) \end{aligned}$$

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Also

$$\begin{aligned}\overline{e(z)} &= \overline{e^z}\\ &= \overline{e^{x+iy}}\\ &= \overline{e^x e^{iy}}\\ &= \overline{e^x(cosy + isiny)}\\ &= \overline{e^x}\,\overline{(cosy + isiny)}\\ &= e^x(cosy - isiny)\end{aligned}$$

Thus $e(\overline{z}) = \overline{e(z)}$. In the similar way it is true that $e^{-}\overline{z} = \overline{e^{-}(z)}$. Thus if z is a fixed point of $e(z)$ or $e^{-}(z)$, then $\overline{z}$ is also the fixed point these functions. Indeed for example if $e(z) = z$ then $e(\overline{z}) = \overline{e(z)} = \overline{z}$. Therefore all the fixed points of $e(z)$ and $e^{-}(z)$ occur in conjugate pairs. The following theorem gives more detailed insight about the fixed points of $e(z)$ and $e^{-}(z)$

Theorem 4.7. *[9] The following statements hold true for the exponential functions $e(z) = e^z$ and $e^{-}(z) = -e^z$:*
1) All the fixed points of $e(z)$ and $e^{-}(z)$ occur in conjugate pairs,
2) Two conjugate members of each pair are distinct,
3) Fixed points of $e(z)$ and $e^{-}(z)$ form a denumerable set,
4) $e^{-}(z)$ has a single real valued fixed point.

Proof: Let $z = x + iy$ be a fixed point of $e(z)$ and $e^{-}(z)$. Therefore $e(z) = e^z =$z and $e^{-}(z) = -e^z = z$. These equations together may be written as

$$e^{\pm}(z) = \pm e^z = z \tag{1}$$

The polar form of the fixed point $z = x + iy$ is $z = x + iy = re^{i(\theta+2k\pi)}$, $k \in \mathbb{Z}$, where $r = \sqrt{x^2 + y^2}$, called absolute value or the modulus of z and $\theta = tan^{-1}(y/x)$, called the argument of z. Therefore
$z = x + iy = \sqrt{x^2 + y^2}[cos(\theta + 2k\pi) + isin(\theta + 2k\pi)]$. Thus $x = \sqrt{x^2 + y^2}cos(\theta + 2k\pi)$ and $y = \sqrt{x^2 + y^2}sin(\theta + 2k\pi)$ or $x = rcos\phi$ and $y = rsin\phi$, where $\phi = \theta + 2k\pi$, $k \in \mathbb{Z}$. Equation (1) becomes

$$\begin{aligned}x + iy &= \pm e^{x+iy}\\ &= \pm e^x e^{iy}\\ &= \pm e^x(cosy + isiny)\end{aligned}$$

By equating real and imaginary parts on both sides:

$$\begin{aligned}x &= \pm e^x cosy\\ y &= \pm e^x siny\end{aligned}$$

$\therefore r^2 = x^2 + y^2 = e^{2x}(cos^2y + sin^2y) = e^{2x}$ and
$\theta = tan^{-1}\left[\frac{\pm e^x siny}{\pm e^x cosy}\right] = tan^{-1}(tany) = y$. Thus if $z = x + iy$ is a fixed point of $e^{\pm}(z)$, then x and y must satisfy $x^2 + y^2 = e^{2x}$ and $y/x = tany$. Rewrite these equations as

$$y^2 = e^{2x} - x^2 \tag{2}$$

$$x = \frac{y}{tany} \tag{3}$$

$$sign\ of\ x = \pm sign\ of\ cosy \tag{4}$$

These three constraints on x and y give all the fixed points of $e(z)$ and $e^{-}(z)$. In order to plot the graphs, we consider $y^2 = f(x)$, where $f(x) = e^{2x} - x^2$ and $x = g(y)$, where $g(y) = \frac{y}{tany}$. Parametric representation of the first function is $x = t, y^2 = f(t) = e^{2t} - t^2$ or $y = \pm\sqrt{e^{2t} - t^2}$ and parametric representation of the second function is $y = t$ and $x = g(t) = \frac{t}{tant}$. Now since y is real, $\sqrt{e^{2t} - t^2}$ must be non-negative. That is

$$e^{2x} - x^2 \geq 0$$
$$\therefore e^{2x} \geq x^2$$
$$\therefore e^{x} \geq x$$

which is true for all $x \in \mathbb{R}$. For given such x, y is given by $y = \pm\sqrt{e^{2x} - x^2}$. Thus $z = x + i\sqrt{e^{2x} - x^2}$ and $z = x - i\sqrt{e^{2x} - x^2}$ are fixed points of $e(z)$ and $e^{-}(z)$. From the following figure 2 it is clear that x and y satisfying the conditions (2) and (3), form a denumerable set of pairs. Consider equation (4). When $y \neq 0$, then $e^{\pm}(z) = z$ imply $\pm cosy > 0$. Hence for $e(z) = z$. y must lie in one of the intervals $(2\pi k - \pi/2, 2\pi k + \pi/2)$. In figure, green dots, eliminates half of the intersections between the functions $f(x)$ and $g(y)$. The remaining intersections (red dots) are those with y in the interval $(2\pi k + \pi/2, 2\pi k + 3\pi/2)$ and mark the fixed points of $e^{-}(z)$.
When $y = 0$, equation (2) imply that $-x = W(1)$ (where $W(1)$ is a Lambert function [[10],[11],[12]]) and equation (4) indicates that it is a fixed point of $e^{-}(z)$ only. It has been marked by the red dot z_0.

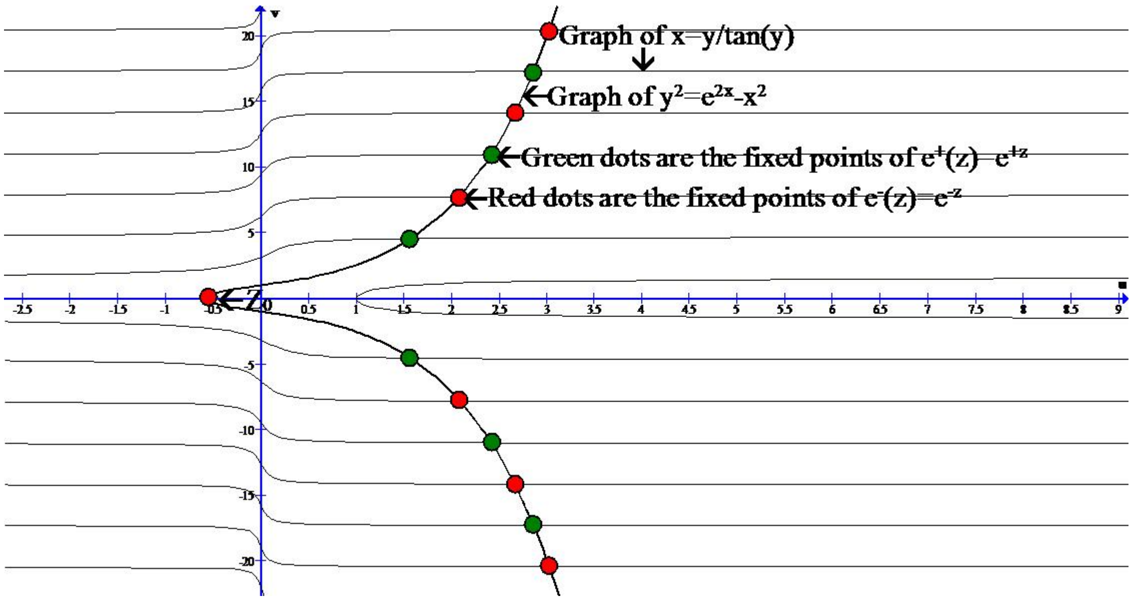

Figure 2: [9]Fixed points of the functions $e^{\pm}(z) = \pm e^{z}$ form a denumerable set of fixed points.

5 Fixed Points in Linear Algebra

In the present section Perron Frobenius theorem and following interesting applications of it are considered:
1) Probability Theory,
2) Dynamical Systems,
3) Economics,
4) Demography by Leslie's Age Distribution Model,
5) Mathematics Behind Internet Search Engines Like Google,
6) Ranking of Professional tennis players.
Brower fixed point theorem is the key to establish Perron Frobenius theorem. We begin with the basic definition of closed unit ball in $\mathbb{R}^n$.

Definition 5.1. *Closed unit ball B^n in $\mathbb{R}^n$ is defined as the set*
$\mathcal{B}^n = \{(u_1, u_2, \cdots, u_n)/u_i \in \mathbb{R}, 1 \leq i \leq n, \|u\| = (u_1^2 + u_2^2 + \cdots + u_n^2)^{1/2} \leq 1\}$

Brower fixed point theorem is stated as follows:

Theorem 5.2. *[8] Any continuous map from the closed unit ball in $\mathbb{R}^n$ to itself has a fixed point.*

The following notions are used in Perron Frobenius theorem.

Definition 5.3. *A $n\times n$ matrix $A=(a_{ij})$ is said to be non negative (respectively positive) and we write $A\geq 0$ (respectively $A>0$) if $a_{ij}\geq 0$ (respectively $a_{ij}>0$) for all i,j.*

Definition 5.4. *A vector X in $\mathbb{R}^n$ is called a non-negative (respectively positive) and written as $X\geq 0$ (respectively $X>0$) if X is a non-negative (respectively positive) when regarded as a matrix.*

Definition 5.5. *A non zero vector X is called an eigen vector of an $n\times n$ matrix A if $AX=\lambda X$ where $\lambda\in\mathbb{R}$. λ is called an eigen value of the matrix A.*

Definition 5.6. *The set of all eigenvalues of a matrix is called its spectrum.*

Definition 5.7. *The largest value of the modulus of an eigenvalue of a matrix A is called the spectrum radius of A. It is denoted by $r(A)$.*

The following lemma is useful.

Lemma 5.8. *If a matrix $A>0$ and u is an eigenvector of A with $u\geq 0$, then $u>0$.*

The statement and proof of Perron Frobenius theorem is as follows:

Theorem 5.9. *[8] Let $A=(a_{ij})$ be a real strictly positive $n\times n$ matrix. Then*
1. A has a positive eigenvalue λ with $r(A)=\lambda$.
2. λ is unique such eigenvalue.
3. The corresponding eigenvector is also strictly positive.
4. A has no other non-negative eigenvector. That is all other eigenvectors are not non-negative.

Proof of (1) and (2): Let S_n be the unit sphere with the center origin in $\mathbb{R}^n$. Define

$$S=\{u=(u_1,u_2,u_3,\cdots,u_n)/\|u\|=1,u_i\geq 0,\forall i=1,2,3,\cdots,n\}$$

Then S is homeomorphic to the closed unit ball $\mathcal{B}^{n-1}$ in $\mathbb{R}^{n-1}$. Define a function $T:S\to S$ by

$$Tu=\frac{Au}{\|Au\|}$$

This function is clearly continuous. So one can apply Brouwer fixed point theorem on T to get a fixed point of T, say $u_0=(u_{0,1},u_{0,2},u_{0,3},\cdots,u_{0,n})$. Thus

$$Tu_0=\frac{Au_0}{\|Au_0\|}=u_0.$$

Let $\lambda=\|Au_0\|$. So that $Au_0=\lambda u_0$. Thus λ is a eigenvalue of the matrix A. Clearly $\lambda>0$.
Proof of (3): By the definition of the set S, all the components of u_0 are non-negative

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and $A > 0$. Thus $Au_0 > 0$. Hence the eigenvector $u_0 > 0$ (see lemma 5.8).
Proof of (4): Next we show that λ has no other eigenvector. This we show by contradiction method. Suppose that there is another eigenvector $u_0' = (u_{0,1}', u_{0,2}', u_{0,3}', \cdots, u_{0,n}')$, independent of u_0. As we know $A > 0$ and $\lambda > 0$, by the similar argument as in the proof of (3) above, we get $u_0' > 0$. Let

$$t = \min_i \left\{ \frac{u_{0,i}}{u_{0,i}'} \right\} = \frac{u_{0,k}}{u_{0,k}'}$$

for some k.
Consider the vector $u_0'' = u_0 - tu_0'$. Then we have

$$\begin{aligned} Au_0'' &= A(u_0 - tu_0') \\ &= Au_0 - tAu_0' \\ &= \lambda u_0 - t\lambda u_0' \\ &= \lambda(u_0 - tu_0') \\ &= \lambda u_0'' \end{aligned}$$

Therefore u_0'' is also an eigenvector of A corresponding to the eigenvalue λ. But

$$\begin{aligned} u_k'' &= u_{0,k} - tu_{0,k}' \\ &= u_{0,k} - \frac{u_{0,k}}{u_{0,k}'} u_{0,k}' \\ &= 0. \end{aligned}$$

This is a contradiction to the fact that u_0'' is an eigenvector of A with the eigenvalue λ, since u_0 and u_0' are linear independent. Thus A has no other non-negative eigenvector. The proof is now complete.

Definition 5.10. *[8] A $n \times n$ matrix is called Markov matrix (Stochastic matrix) if it is non-negative and the sum of the elements of each column is 1.*

Theorem 5.11. *[8] A Markov matrix always has an eigenvalue 1 and all other eigenvalues are in absolute value smaller than or equal to 1.*

As stated in the beginning, Perron Frobenius theorem 5.9 and the above theorem 5.11 are important in computers and all other applications mentioned there. One of the interesting applications is considered below.

Example 5.12. *[8] Working of the internet search engine Google is as follows:*
1) Google first explore the web and locate the web pages which have public access.
2) Google search engine then give the index to the data found in the above step.
3) Finally the engine rate the importance of each page in the data on the probability basis, which leads to more important page to appear on the computer screen.

Step 3 is considered below in detail. Google search engine is based on the system of "PageRank" in order to prioritize the importance of a web page. The ideas behind the concept of PageRank are as follows:
1) A page on internet is linked to another page. An example of this situation can be given as a Facebook page has a link that takes the user to a Twitter page. This is as if one page votes for another page.
2) A web page is ranked more important than the other if it is linked by more number of web pages. That is a page is ranked higher if more pages votes for it.
3) Moreover one vote becomes more important than the other if it is obtained from more important page.
The above description is explained further by taking a particular case. Let the web page P has pages $V_1, V_2, V_3, \cdots, V_n$ *as its' voter pages. Let* $V(P)$ *be the number of votes given by page P to other web pages.*

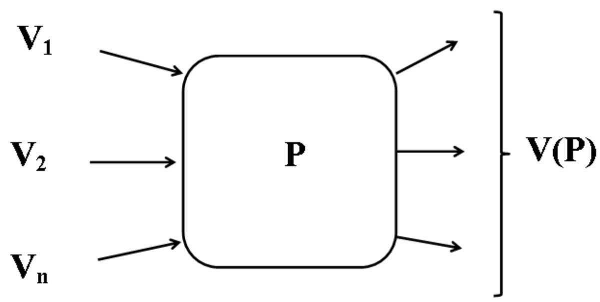

Figure 3: Votes casted to and casted by the web page P

Further suppose the web page Vj casts n_j *votes to other pages and out of* n_j *votes only one is casted for the page P. Then it increases the importance of the page* P *by* $\frac{PR(V_j)}{n_j}$ *. Here PR(Vj) is the PageRank of the page Vj. Assume that the internet has m pages* $P_1, P_2, P_3, \cdots, P_m$ *. Let* N_k *be the number of outgoing links from Pk. Then*

$$PR(P_k) = \sum_{j \neq k} \frac{PR(P_j)}{N_j}, k = 1, 2, 3, \cdots, m$$

We also note that
1) Different PageRank form a discrete probability distribution (or probability density function) over the web pages. Hence Sum of all PageRanks = 1,
2) $PR(P_k)$ *corresponds to the principal eigenvector of the normalized link matrix of the web. Therefore it can be calculated from above algorithm.*
Suppose a web page contains four pages P_1, P_2, P_3, P_4 *. The links from one page to other are given in the following figure.*

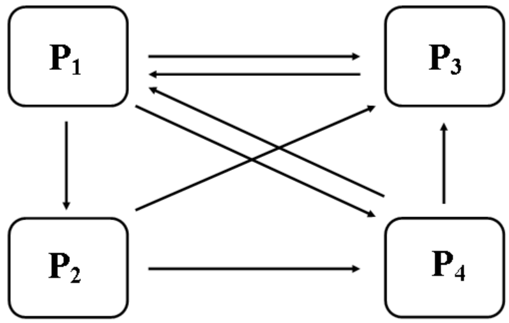

Figure 4: Arrows shows the links between the web pages

Let $PR(P_1) = x, PR(P_2) = y, PR(P_3) = z, PR(P_4) = w$ *. Then we get* $x = \frac{z}{1} + \frac{w}{2}, y = \frac{x}{3}, z = \frac{x}{3} + \frac{y}{2} + \frac{w}{2}, w = \frac{x}{3} + \frac{y}{1}$ *. Therefore we consider the vector*

$$X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

We have the system $AX = X$ *, with*

$$\begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Here A *has eigenvalue 1 and its unique corresponding eigenvector from Perron-Frobenius theorem* **??** *is*

$$X = \begin{bmatrix} 12 \\ 4 \\ 9 \\ 6 \end{bmatrix}$$

Normalizing in order to have stochastic vector we get

$$X = \frac{1}{31} \begin{bmatrix} 12 \\ 4 \\ 9 \\ 6 \end{bmatrix}$$

This ranking makes the page P_1 *the most important web page.*

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6 Fixed Points in Transformational Geometry

Definition 6.1. *Isometry is a distance and angle preserving transformation.*

An isometry also preserves shapes.

Definition 6.2. *The fixed points of an isometry are those points in the plane whose images are themselves.*

Remark 6.3. *The number of fixed points of isometries helps us to characterize all the isometries. Various types of isometries and their characterization by the number of fixed points is shown by the following table.*

Table 1: Table showing the number of fixed points of an isometry geometrical transformation

Sl. No.	Isometry	Definition	Number of fixed points
1	Translation	Under translation a point $A(x, y)$ is translated to the point $B(x + a, y + b)$.	No fixed points or all the points are fixed if $a = 0, b = 0$. In this case translation becomes an identity translation.
2	Rotation	Under rotation about an angle θ a point $A(x, y)$ is translated to the point $B(x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$.	Rotation has unique fixed point that is its point of rotation.
3	Reflection	Under the reflection over X axis a point $A(x, y)$ is translated to the point $B(x, -y)$, and under the reflection over Y axis point $A(x, y)$ is translated to the point $B(-x, y)$.	All the points on the line of reflection are fixed.

7 Conclusion

This research article demonstrates that the fixed points are very useful in Mathematics. Analysis of the mappings and their fixed points enable us to successfully handle many situations in pure and applied Mathematics. Main branches of applied Mathematics like linear algebra, numerical analysis and branches of pure Mathematics like real analysis, complex analysis, number theory utilize fixed points and related results. So it is essential to explore more on fixed points.

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